Extreme Points of Matrix Convex Sets

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Free Spectrahedra

LMIs and Spectrahedra

Notation: Let $SM_n(\mathbb{R}) \subseteq \mathbb{R}^{n\times n}$ denote real symmetric $n \times n$ matrices and let $SM_n(\mathbb{R})^g$ denote g-tuples of elements of $SM_n(\mathbb{R})$.

A (monic) linear pencil is a matrix valued function L_A of the form

$$L_A(x) := I_d - \sum_{j=1}^{g} A_j x_j,$$

where $A = (A_1, A_2, \ldots, A_g) \in SM_d(\mathbb{R})^g$ is a $g-$tuple of real symmetric $d \times d$ matrices and $x := \{x_1, \cdots, x_g\} \in \mathbb{R}^g$.

A Linear Matrix Inequality (LMI) is one of the form:

$$L_A(x) \succeq 0.$$

The set of solutions

$$\mathscr{S}_A := \{(x_1, x_2, \cdots, x_g) \in \mathbb{R}^g : I_d - \sum_{j=1}^{g} A_j x_j \quad \text{is PSD}\}$$

is a convex set called a spectrahedron.

Evaluation on Matrix Variables: Free Spectrahedra

For a g-tuple $X = (X_1, \ldots, X_g)$ of $n \times n$ real symmetric matrices the evaluation of L_A on X is

$$L_A(X) := I_{dn} - \sum_{j=1}^{g} A_j \otimes X_j.$$

Here

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes X = \begin{pmatrix} 1X & 2X \\ 3X & 4X \end{pmatrix}.$$

For each fixed n define the solution set

$$\mathcal{D}_A(n) = \{X \in SM_n(\mathbb{R})^g : I_{dn} - \sum_{j=1}^{g} A_j \otimes X_j \quad \text{is PSD}\}$$

The set $\mathcal{D}_A = \cup_n \mathcal{D}_A(n) \subset \cup_n SM_n(\mathbb{R})^g$ is called a free spectrahedron.

Classical Convexity and the Convex Hull

Let V be a vector space and let $K \subseteq V$. A convex combination of elements of K is a sum of the form $\lambda x_1 + (1-\lambda)x_2$ where $x_1, x_2 \in K$ and $\lambda \in [0,1]$.

The convex hull of K denoted $\mathrm{conv}(K)$ is the set of all convex combinations of elements of K. The set K is convex if $\mathrm{conv}(K) = K$.

A point $x \in K$ is an extreme point of K if $x = \lambda x_1 + (1-\lambda)x_2$ for $x_1, x_2 \in K$ and $\lambda \in (0,1)$ implies $x = x_1 = x_2$

Theorem [Krein-Milman]

Let K be a compact convex set. Then K is the convex hull of its extreme points. Additionally, if $S \subseteq K$ satisfies $\mathrm{conv}(S) = K$ then S contains the extreme points of K.

Matrix Convex Sets and The Matrix Convex Hull

Given a set $K \subseteq \cup_n SM_n(\mathbb{R})^g$, a matrix convex combination is a finite sum of the form

$$\sum_{\ell=1}^{k} V_\ell^T X^\ell V_\ell \in SM_n(\mathbb{R})^g \qquad \sum_{\ell=1}^{k} V_\ell^T V_\ell = I_n$$

where $X^\ell = (X_1^\ell, \ldots, X_g^\ell)$ is a g-tuple of $n_\ell \times n_\ell$ symmetric matrices and the V_ℓ are $n_\ell \times n$ matrices.

A set is matrix convex if it is closed under matrix convex combinations.

Remark

Free Spectrahedra are Matrix Convex. Furthermore, every matrix convex set is a (possibly infinite) intersection of free spectrahedra (Effros-Winkler 97).

The matrix convex hull of K, denoted $\mathrm{co}^{\mathrm{mat}}(K)$, is the set of all matrix convex combinations of elements K.

Operator Theory and Matrix Convex Sets

In 1999 Webster and Winkler showed that there is a one to one correspondence between compact matrix convex sets and sets of unital completely positive maps on an operator system.

Theorem [Webster-Winkler 99]

Let K be a compact matrix convex set. Then K is matrix affine homeomorphic to the set of unital completely positive maps on the operator system $A(K)$ of matrix affine maps on K.

In addition, if $\mathcal{R}$ is an operator system then the set of u.c.p maps on $\mathcal{R}$ is a compact matrix convex set.

Matrix Convex Sets: Basic Definitions

Given a matrix convex set $K \subset \cup_n SM_n(\mathbb{R})^g$ and a fixed integer n, define the set K at level n, denoted $K(n)$, by

$$K(n) = K \cap SM_n(\mathbb{R})^g.$$

Remark

Then $K(n)$ is convex for each n.

Say K is bounded if there exists a real number $C > 0$ such that $C - \sum_{j=1}^{g} X_j^2 \succeq 0$ for all $X = (X_1, \ldots, X_g) \in K$.

Say K is closed if $K(n)$ is closed for each $n \in \mathbb{N}$. We will say K is compact if K is closed and bounded.

Irreducibility

Let $X = (X_1, \ldots, X_g) \in SM_n(\mathbb{R})^g$. A subspace $\mathscr{H} \subset \mathbb{R}^n$ is a **reducing subspace** for X if $\mathscr{H}$ and $\mathscr{H}^\perp$ are both invariant subspaces of X_i for $i = 1, \ldots, g$. Say X is **irreducible** if X has no reducing subspaces.

Equivalently, X is irreducible if X is not unitarily equivalent to a direct sum.

For example $X = Y \oplus Z$ is not irreducible.

Extreme Points

Classical Convexity of Free Spectrahedra

Matrix convex sets have three types of extreme points: Euclidean (classical), Matrix, and Free extreme points.

$X \in K(n)$ is a Euclidean extreme point of K if X is a classical extreme point of $K(n)$. The matrix extreme points of K are the tuples which correspond to pure u.c.p maps.

Matrix and Euclidean extreme points are known to span (Webster-Winkler 99), but there are TOO MANY.

Free extreme points are the most restricted extreme points. They are closely related to the classical Arveson boundary.

Free Extreme Points

Say $X \in K(n)$ is a **free extreme point** of K if

$$X = \sum_{\ell=1}^{k} V_\ell^T Y^\ell V_\ell \qquad \sum_{\ell=1}^{k} V_\ell^T V_\ell = I_n$$

where each $Y^\ell = (Y_1^\ell, \ldots, Y_g^\ell) \in K(n_\ell)$ is irreducible and $0 \neq V_\ell : \mathbb{R}^n \to \mathbb{R}^{n_\ell}$, then for each ℓ we have $n_\ell = n$ and X is unitarily equivalent to Y^ℓ.

Let $\partial^{\mathrm{free}} K$ denote the set of free extreme points of K.

Free Extreme Points

Lemma [E-Helton-Klep-McCullough 17]

Let K be a matrix convex set. If X is a free extreme point of K then X is a matrix extreme point of K. Also, if X is a matrix extreme point of K then X is a Euclidean extreme point of K. In notation

$$\partial^{\mathrm{free}} K \subset \partial^{\mathrm{mat}} K \subset \partial^{\mathrm{euc}} K.$$

Free extreme points are computationally efficient.

Determining if $X \in \mathcal{D}_A(n)$ is a free extreme point of the free spectrahedron $\mathcal{D}_A$ is equivalent to solving a linear system in ng unknowns and $d(\dim \ker L_A(X))$ constraints.

Determining if X is a Euclidean extreme point of $\mathcal{D}_A$ is a linear system in $n(n+1)g/2$ unknowns $dn(\dim \ker L_A(X))$ constraints. There is no known algorithm to determine if X is matrix extreme.

Free Extreme Points

Question

Let $\boldsymbol{K}$ be a compact matrix convex set. Then do the finite dimensional free extreme points of $\boldsymbol{K}$ span $\boldsymbol{K}$ though matrix convex combinations?

We will show

Theorem [E-Helton 19]

If $\boldsymbol{K} = \mathcal{D}_{\boldsymbol{A}}$ is a free spectrahedron then YES, $\mathcal{D}_{\boldsymbol{A}}$ is the matrix convex hull of its finite dimensional free extreme points. Furthermore the free extreme points are the minimal spanning set.

Theorem [E 18]

There are examples of compact matrix convex sets (which are not free spectrahedra) which have NO FINITE DIMENSIONAL ABSOLUTE EXTREME POINTS.

Operator Theory History

In 1969 Arveson conjectures free extreme span in infinite dimensional setting of “operator convex sets” (formulated in terms of u.c.p maps).

Dritschel McCullough 2005 - proved true if BIG cardinality’s allowed, e.g. free extreme points are operators on a nonseperable Hilbert space.

Theorem [Davidson-Kennedy 2015]

Any element $\boldsymbol{X}$ of a compact operator convex set is an “operator” convex combination of (infinite dimensional) free extreme points which have the same cardinality as $\boldsymbol{X}$.

Davidson-Kennedy settled infinite dimensional setting, but finite dimensions remained open.

Matrix Convex Combinations as Dilations

Dilations

Let K be a matrix convex set and let $X \in K(n)$. An ℓ-dilation of X is a g-tuple of the form

$$Y = \begin{pmatrix} X & \beta \\ \beta^T & \gamma \end{pmatrix} \in SM_{n+\ell}(\mathbb{R})^g$$

where β is a g-tuple of $n \times \ell$ matrices and $\gamma \in SM_\ell(\mathbb{R})^g$.

Note that Y can be an element of K.

Dilations as Matrix Convex Combinations

Matrix convex combinations can be expressed in terms of isometric contractions of dilations.

Let

$$X = \sum_{\ell=1}^{k} V_\ell^T Y^\ell V_\ell \in SM_n(\mathbb{R})^g \qquad \sum_{\ell=1}^{k} V_\ell^T V_\ell = I_n$$

be a matrix convex combination of elements of K. Set

$$Y = \oplus_{\ell=1}^{k} Y^\ell \qquad V^T = \begin{pmatrix} V_1^T & \cdots & V_k^T \end{pmatrix}.$$

Then Y is an element of K and

$$X = V^T Y V \text{ and } V^T V = I_n.$$

Note that Y is unitarily equivalent to a dilation of X.

Dilations and Extreme Points

Let K be a matrix convex set and let $X \in K(n)$. Say X is an Arveson extreme point of K if

$$Z = \begin{pmatrix} X & \beta \\ \beta^T & \gamma \end{pmatrix}$$

is an element of K implies $\beta = 0$.

Theorem [E-Helton-Klep-McCullough 17]

Let K is a matrix convex set, n is a positive integer and $X \in K(n)$. Then X is unitarily equivalent to a direct sum of free extreme points of K if and only if X is an Arveson extreme point of K.

A Free Caratheodory Theorem

Theorem [E-Helton 19]

Let $\mathcal{D}_A$ be a compact free spectrahedron and let $X \in \mathcal{D}_A(n)$. Then there is a dilation Z of X which has size less than or equal to $n(g+1)$ such that Z is an Arveson extreme point.

Equivalently X is a matrix convex combination of free extreme points of $\mathcal{D}_A$ whose sum of sizes is less than or equal to $n(g+1)$.

In contrast the bound on the number of classical extreme points needed in the Caratheodory Theorem is $n(n+1)g/2+1$

Although fewer free extreme points are needed, we may pay a price on the size of points required.

Matrix Convex Sets with no Free Extreme Points

NC polygons

Let $\mathcal{H} = \ell^2(\mathbb{N})$ and for each positive integer n identify the subspace $\ell^2(\{1,\ldots,n\})$ of $\mathcal{H}$ with $\mathbb{R}^n$. Let $\mathbb{S}(\mathcal{H})^g$ denote g-tuples of symmetric operators on $\mathcal{H}$.

For a finite collect $\mathcal{Z} = \{Z^1,\ldots,Z^k\} \subset \mathbb{S}(\mathcal{H})^g$ define

$$\begin{aligned}\mathrm{hull}^{\mathrm{nc}}(\mathcal{Z})(n) \;=\; & \{X \in SM_n(\mathbb{R})^g : \; X = \textstyle\sum_{\ell=1}^{\infty} V_\ell^T Z^{j_\ell} V_\ell \text{ where} \\ & \textstyle\sum_{\ell=1}^{\infty} V_\ell^T V_\ell = I_n \text{ and } V_\ell : \mathbb{R}^n \to \mathcal{H}\}.\end{aligned}$$

The noncommutative (nc) convex hull of $\mathcal{Z}$ is the set

$$\mathrm{hull}^{\mathrm{nc}}(\mathcal{Z}) = \cup_n \mathrm{hull}^{\mathrm{nc}}(\mathcal{Z})(n).$$

A noncommutative polygon with corners $Z^1,\ldots,Z^k$ is the nc convex hull of $\{Z^1,\ldots,Z^k\}$ where each Z^ℓ is an irreducible g-tuple of symmetric operators on $\mathcal{H}$.

NC polygons

Theorem [E 18]

Let $\mathbf{hull^{nc}}(\mathcal{Z})$ *be a nc polygon with corners* $\boldsymbol{Z^1, \ldots, Z^k}$ *which are irreducible* $\boldsymbol{g}$*-tuples of compact operators on* $\mathcal{H}$*. Assume* $\mathbf{0}$ *is in the convex hull of the joint numerical range of* $\bigoplus_{\ell=1}^{k} \boldsymbol{Z^\ell}$*. Then* $\mathbf{hull^{nc}}(\mathcal{Z})$ *is a compact matrix convex set that has no free extreme points.*

Remark

There are also examples of compact matrix convex sets (which are not free spectrahedra) that are the span of their free extreme points.

Summary

1. **Let $\mathcal{D}_A$ be a compact free spectrahedron. Then $\mathcal{D}_A$ is the matrix convex hull of its finite dimensional free extreme point.**

2. **If $X \in \mathcal{D}_A$ is a g-tuple of $n \times n$ symmetric matrices then X dilates to an Arveson extreme point Z of $\mathcal{D}_A$ which has size less than or equal to $n(g+1)$.**

3. **Arveson dilations of an element of a compact free spectrahedron can be constructed by taking a sequence of maximal 1-dilations.**

Summary cont.

4. **Each maximal 1-dilation can be computed by solving a semidefinite program followed by a local maximization of a convex quadratic over a spectrahedron.**

5. **If K is a noncommutative polygon whose corners have no finite dimensional reducing subspaces and 0 is in the finite interior of K then K is a compact matrix convex set which has no free extreme points.**

Free Extreme Points of Free Spectrahedra: Proofs

The Dilation Subspace

We now focus on the case where our matrix convex set is a free spectrahedron.

Let $\mathcal{D}_A$ be a free spectrahedron and let $X \in \mathcal{D}_A$. The dilation subspace of $\mathcal{D}_A$ at X, denoted $\mathfrak{K}_{A,X}$ is the subspace

$$\mathfrak{K}_{A,X} = \{\beta \in (\mathbb{R}^{n\times 1})^g |\ \ker L_A(X) \subset \ker \Lambda_A(\beta)\}$$

where $\Lambda_A(\beta) = \sum_{\ell=1}^{g} A_\ell \otimes \beta_\ell$.

Roughly speaking, the dilation subspace is the linear span of tuples $\beta \in (\mathbb{R}^{n\times 1})^g$ which can dilate X to an element of $\mathcal{D}_A$.

The Dilation Subspace

Lemma [E-Helton]

Let $\mathcal{D}_A$ be a free spectrahedron and let $X \in \mathcal{D}_A(n)$.

1. *Let $\beta \in (\mathbb{R}^{n\times 1})^g$. Then $\beta \in \mathfrak{K}_{A,X}$ if and only if there is a real number $c > 0$ and a tuple $\gamma \in \mathbb{R}^g$ such that*

$$Y = \begin{pmatrix} X & c\beta \\ c\beta^T & \gamma \end{pmatrix} \in \mathcal{D}_A(n+1).$$

   In particular, if

$$Y = \begin{pmatrix} X & \beta \\ \beta^T & \gamma \end{pmatrix} \in \mathcal{D}_A(n+1)$$

   then $\beta \in \mathfrak{K}_{A,X}$.

2. *X is (unitarily equivalent to) a direct sum of free extreme points of $\mathcal{D}_A$ if and only if $\dim \mathfrak{K}_{A,X} = 0$.*

Partial Proof of Lemma

Conjugating by permutation matrices (often called the canonical shuffles) shows

$$L_A \begin{pmatrix} X & c\beta \\ c\beta^T & \gamma \end{pmatrix} \succeq 0 \text{ if and only if } \begin{pmatrix} L_A(X) & -c\Lambda_A(\beta) \\ -c\Lambda_A(\beta^T) & L_A(\gamma) \end{pmatrix} \succeq 0$$

Taking the appropriate Schur compliment shows $L_A(Y) \succeq 0$ if and only if $L_A(\gamma) \succeq 0$ and

$$L_A(X) - c^2\Lambda_A(\beta)L_A(\gamma)^\dagger\Lambda_A(\beta^T) \succeq 0.$$

This shows that $Y \in \mathcal{D}_A$ implies $\ker L_A(X) \subset \ker \Lambda_A(\beta)$. That is, $\beta \in \mathfrak{K}_{A,X}$.

Maximal $\mathbf{1}$-dilations

Goal: Given a free spectrahedron $\mathcal{D}_A$ and $X \in \mathcal{D}_A$, construct a dilation of X which is a direct sum of free extreme points by taking a sequence of dilations which reduce the dimension of the dilation subspace.

Maximal 1-dilations

Given a free spectrahedron $\mathcal{D}_A$ and a tuple $X \in \mathcal{D}_A(n)$, say the dilation

$$Y = \begin{pmatrix} X & \hat{c}\hat{\beta} \\ \hat{c}\hat{\beta}^T & \hat{\gamma} \end{pmatrix} \in \mathcal{D}_A(n+1)$$

is a maximal 1-dilation of X if Y is a 1-dilation of X and $\hat{\beta}$ is a nonzero element of $\mathfrak{K}_{A,X}$ and the real number $\hat{c}$ and tuple $\hat{\gamma} \in \mathbb{R}^g$ are solutions to the sequence of maximization problems

$$\begin{aligned} \hat{c} := & \ \text{Maximizer}_{c\in\mathbb{R},\gamma\in\mathbb{R}^g} \ c \\ \text{s.t.} & \ L_A \begin{pmatrix} X & c\hat{\beta} \\ c\hat{\beta}^T & \gamma \end{pmatrix} \succeq 0 \end{aligned}$$

$$\begin{aligned} \text{and} \quad \hat{\gamma} := & \ \text{Local Maximizer}_{\gamma\in\mathbb{R}^g} \ \|\gamma\| \\ \text{s.t.} & \ L_A \begin{pmatrix} X & \hat{c}\hat{\beta} \\ \hat{c}\hat{\beta}^T & \gamma \end{pmatrix} \succeq 0 \end{aligned}$$

where $\|\cdot\|$ denotes the usual norm on $\mathbb{R}^g$.

Maximal 1-dilations dilate toward free extreme points

Theorem [E-Helton 19]

Let $\mathcal{D}_A$ be a compact free spectrahedron and let $X \in \mathcal{D}_A(n)$. Assume X is not (unitarily equivalent to) a direct sum of free extreme points of $\mathcal{D}_A$. Then there exists a nontrivial maximal 1-dilation $Y \in \mathcal{D}_A(n+1)$ of X. Furthermore,

$$\dim \mathfrak{K}_{A,Y} < \dim \mathfrak{K}_{A,X}.$$

Writing a X as a matrix convex combination of free extreme points is computationally feasible and can be accomplished by computing a sequence of maximal 1-dilations of X.

Thank You!

Thank you for coming!