Inclusion constants for matrix convex sets relevant to quantum incompatibility

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Convex Combinations

Given a set $C \subset \mathbb{R}^g$ and a finite collection of tuples $\{x^\ell\} \subset C$ where $x^\ell = (x_1^\ell, x_2^\ell, \ldots, x_g^\ell)$ and coefficients $\alpha_\ell \geq 0$ a convex combination is a sum of the form

$$\sum_{\ell=1}^{k} \alpha_\ell x^\ell \in \mathbb{R}^g \qquad \text{such that} \qquad \sum_{\ell=1}^{k} \alpha_\ell = 1$$

The convex hull of a set C is the set of all convex combinations of C. Say C is convex if it is closed under convex combinations.

A point $x \in C$ is an extreme point of C if it cannot be expressed as a nontrivial convex combination of elements of C.

Convex sets have many nice properties

Theorem [Carathéodory (also see Krein-Milman)]

Let $C \subset \mathbb{R}^g$ be a closed bounded convex set. Then C is the convex hull of its extreme points.

Furthermore, every element of C can be expressed as a convex combination of at most $g+1$ extreme points of C.

Linear matrix inequalities give convex sets

A (monic) linear pencil is a matrix valued function $L_{\mathcal{A}}$ of the form

$$L_{\mathcal{A}}(x) := \mathbf{I}_d - \sum_{j=1}^{g} \mathbf{A}_j x_j = I_d - \Lambda_{\mathcal{A}}(x),$$

where $\mathcal{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_g)$ with each $\mathbf{A}_j$ symmetric $d \times d$ and $x = (x_1, \ldots, x_g) \in \mathbb{R}^g$

A Linear Matrix Inequality (LMI) is one of the form:

$$L_{\mathcal{A}}(x) \succeq 0, \qquad i.e., \qquad L_{\mathcal{A}}(x) \text{ is positive semidefinite.}$$

The set of solutions x above is a convex set called a spectrahedron. Spectrahedra are the feasibility domains of convex optimization problems called semidefinite programs (SDP).

Spectrahedron example

Take $\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$L_{\mathcal{A}}(x) = \mathbf{I}_2 - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x_1 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_2 = \begin{pmatrix} 1 - x_1 & -x_2 \\ -x_2 & 1 + x_1 \end{pmatrix}$$

Observe $L_{\mathcal{A}}(x) \succeq 0$ IFF $\det(L_{\mathcal{A}}(x)) = 1 - x_1^2 - x_2^2 \geq 0$. So $L_{\mathcal{A}}(x)$ defines circle in $\mathbb{R}^2$.

Dimension free sets

Let $SM_n(\mathbb{R})^g$ denote g-tuples of real symmetric $n \times n$ matrices. I.e. if $\mathcal{X} \in SM_n(\mathbb{R})^g$ then

$$\mathcal{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_g)$$

where each $\mathbf{X}_i$ is a symmetric $n \times n$ matrix.

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Define $SM(\mathbb{R})^g = \cup_{n=1}^{\infty} SM_n(\mathbb{R})^g$. A subset of $SM(\mathbb{R})^g$ is a dimension free set.

Our goal: Study solution sets of linear matrix inequalities over $SM(\mathbb{R})^g$.

Free Linear matrix inequalities

A free (monic) linear pencil is a matrix valued function $L_{\mathcal{A}}$ of the form

$$L_{\mathcal{A}}(\mathcal{X}) := \mathbf{I}_{dn} - \sum_{j=1}^{g} \mathbf{A}_j \otimes \mathbf{X}_j = \mathbf{I}_{dn} - \Lambda_{\mathcal{A}}(\mathcal{X}),$$

where $\mathcal{A} \in SM_d(\mathbb{R})^g$ and $\mathcal{X} \in SM_n(\mathbb{R})^g$. Here $\otimes$ denotes the Kronecker Product. E.g.

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 2\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} & 3\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} \\ 3\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} & 4\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} \end{pmatrix}$$

A Free Linear Matrix Inequality (LMI) is one of the form:

$$L_{\mathcal{A}}(\mathcal{X}) \succeq 0.$$

Free spectrahedra

For each fixed n the solution set

$$\mathcal{D}_{\mathcal{A}}(n) = \{\mathcal{X} \in SM_n(\mathbb{R})^g : L_{\mathcal{A}}(\mathcal{X}) = \mathbf{I}_{dn} - \sum_{j=1}^{g} \mathbf{A}_j \otimes \mathbf{X}_j \succeq 0\}$$

is called a free spectrahedron at level n.

The set $\mathcal{D}_{\mathcal{A}} = \cup_n \mathcal{D}_{\mathcal{A}}(n) \subset \cup_n SM_n(\mathbb{R})^g$ is called a free spectrahedron.

If $\mathcal{A}$ a tuple of simultaneously diagonalizable matrices, then $\mathcal{D}_{\mathcal{A}}$ is called free polyhedron.

Matrix Convex Combinations

Given a finite collection of tuples $\{\mathcal{X}^\ell\} \subset SM(\mathbb{R})^g$ where $\mathcal{X}^\ell = (\mathbf{X}_1^\ell, \ldots, \mathbf{X}_g^\ell) \in SM_{n_\ell}(\mathbb{R})^g$, a matrix convex combination is a sum of the form

$$\sum_{\ell=1}^{k} \mathbf{V}_\ell^T \mathcal{X}^\ell \mathbf{V}_\ell \in SM_n(\mathbb{R})^g \qquad \text{such that} \qquad \sum_{\ell=1}^{k} \mathbf{V}_\ell^T \mathbf{V}_\ell = \mathbf{I}_n$$

Here the $\mathbf{V}_\ell$ are $n_\ell \times n$ matrices which serve as convex coefficients, and

$$\mathbf{V}_\ell^T \mathcal{X}^\ell \mathbf{V}_\ell = (\mathbf{V}_\ell^T \mathbf{X}_1^\ell \mathbf{V}_\ell, \ldots, \mathbf{V}_\ell^T \mathbf{X}_g^\ell \mathbf{V}_\ell).$$

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For $K \subset SM(\mathbb{R})^g$ let $\mathrm{co}^{\mathrm{mat}}(K)$ denote the set of matrix convex combinations of K. Say K is matrix convex if it is closed under matrix convex combinations, i.e., if $K = \mathrm{co}^{\mathrm{mat}}(K)$.

Say K is bounded if there exists a $M \geq 0$ such that $M\mathbf{I} - \sum_{i=1}^{g} \mathbf{X}_i^2 \succeq 0$ for all $\mathcal{X} = (\mathbf{X}_1, \dots, \mathbf{X}_g) \in K$.

Matrix convex combinations allow for convex combinations of tuples of different sizes

For example, if $\mathcal{X}^1 \in SM_{n_1}(\mathbb{R})^g$ and $\mathcal{X}^2 \in SM_{n_2}(\mathbb{R})^g$ and

$$\mathbf{V}_1^T = (\mathbf{I}_{n_1} \quad \mathbf{0}_{n_1 \times n_2}) \qquad \text{and} \qquad \mathbf{V}_2^T = (\mathbf{0}_{n_2 \times n_1} \quad \mathbf{I}_{n_2}),$$

then

$$\mathbf{V}_1^T \mathcal{X}^1 \mathbf{V}_1 + \mathbf{V}_2^T \mathcal{X}^2 \mathbf{V}_2 = \mathcal{X}^1 \oplus \mathcal{X}^2 = \begin{pmatrix} \mathcal{X}^1 & 0 \\ 0 & \mathcal{X}^2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{V}_1^T \mathbf{V}_1 + \mathbf{V}_2^T \mathbf{V}_2 = \mathbf{I}_{n_1+n_2}.$$

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On the other hand, if $\mathcal{X} \in SM_n(\mathbb{R})^g$, and $\mathbf{V} \in \mathbb{R}^{n \times m}$ and $\mathbf{V}^T \mathbf{V} = \mathbf{I}_m$, then

$$\mathbf{V}^T \mathcal{X} \mathbf{V} \in SM_m(\mathbb{R})^g$$

is a matrix convex combination of $\mathcal{X}$.

Matrix convex combinations vs dilations

Given a finite collection of tuples $\{\mathcal{X}^\ell\}_{\ell=1}^k \subset SM(\mathbb{R})^g$ and matrices $\mathbf{V}_\ell \in M_{n_\ell \times n}(\mathbb{R})$ such that $\mathbf{V}_\ell^T \mathbf{V}_\ell = \mathbf{I}_n$, define

$$\mathcal{X} = \oplus_{\ell=1}^k \mathcal{X}^\ell \qquad \text{and} \qquad \mathbf{V}^T = (\mathbf{V}_1^T \ \ldots \ \mathbf{V}_k^T).$$

Then

$$\sum_{\ell=1}^k \mathbf{V}_\ell^T \mathcal{X}^\ell \mathbf{V}_\ell = \mathbf{V}^T \mathcal{X} \mathbf{V} \qquad \text{and} \qquad \mathbf{V}^T \mathbf{V} = \mathbf{I}.$$

Sets defined by Free LMI are matrix convex

Free spectrahedra are matrix convex.

Theorem [Helton-McCullough 12]

Let p be a noncommutative polynomial and let $\mathcal{D}_p$ be the component containing 0 of $\{\mathcal{X} \in SM(\mathbb{R})^g | p(X) \succeq 0\}$. If $\mathcal{D}_p$ is matrix convex, then $\mathcal{D}_p$ is a free spectrahedron.

Question: What is the right notion of extreme point for matrix convex sets (and in particular for free spectrahedra)?

Extreme points of matrix convex sets

Say $\mathcal{X}$ is a matrix extreme point of $K \subset SM(\mathbb{R})^g$ if $\mathcal{X}$ cannot be expressed as a nontrivial matrix convex combination of elements of K *which have size less than or equal to* $\mathcal{X}$.

Say $\mathcal{X}$ is a free (absolute) extreme point of $K \subset SM(\mathbb{R})^g$ if $\mathcal{X}$ cannot be expressed as a nontrivial matrix convex combination of any elements of K.

Matrix extreme vs free extreme

Let $K \subset SM(\mathbb{R})^g$ be a (level-wise) closed bounded matrix convex set

Matrix extreme points

1. Always span K through matrix convex combinations. (Webster-Winkler 99)
2. Not necessarily a minimal spanning set.
3. Carathéodory bound: $\mathcal{X} \in K(n)$ can be expressed as a sum of at most $n^2(g+1)$ matrix extreme points of K. (Hartz-Lupini 21)

Free extreme points

1. Can fail to exist. (E 18, Passer 22)
2. Necessarily a minimal spanning set if they span.
3. Carathéodory bound: If K is a free spectrahedron, then $\mathcal{X} \in K(n)$ can be expressed as matrix convex combo of free extreme points of K with sum of sizes at most $n(g+1)$. (E-Helton 19)

Containment of Matrix Convex Sets

Free spectrahedral containment

Let $\mathcal{A} \in SM_{d_1}(\mathbb{R})^g$ and $\mathcal{B} \in SM_{d_2}(\mathbb{R})^g$. Determining the spectrahedral containment $\mathcal{D}_{\mathcal{A}}(1) \subset \mathcal{D}_{\mathcal{B}}(1)$ is NP-hard in general . (Ben-Tal, Nemirovski)

Free spectrahedral containment

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Determining the optimal constant γ such that $\mathcal{D}_{\mathcal{A}} \subset \gamma\mathcal{D}_{\mathcal{B}}$ is a semidefinite program. (Helton, Klep, McCullough).

HKM show that $\mathcal{D}_{\mathcal{A}} \subset \gamma\mathcal{D}_{\mathcal{B}} = \mathcal{D}_{\mathcal{B}/\gamma}$ if and only if the map τ defined by

$$\tau(\mathbf{I}_{d_1}) = \gamma\mathbf{I}_{d_2} \qquad \text{and} \qquad \tau(\mathbf{A}_j) = \mathbf{B}_j \qquad \text{for } j = 1, \ldots, g.$$

is completely positive, which happens if and only if τ is d_2-positive.

Containment of general matrix convex sets and free polar duals

Let K be a compact matrix convex set. Given $\mathcal{X} \in SM(\mathbb{R})^g$, how can one check if $\mathcal{X} \in K$?

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The free polar dual K° of K is

$$K^\circ := \{\mathcal{Y} \in SM(\mathbb{R})^g : \ L_{\mathcal{Z}}(\mathcal{Y}) \succeq 0 \text{ for all } \mathcal{Z} \in K\} = \cap_{\mathcal{Z} \in K} \mathcal{D}_{\mathcal{Z}}$$

A quick check now shows

$$\mathcal{X} \in K \qquad \Longleftrightarrow \qquad K^\circ \subseteq \{\mathcal{X}\}^\circ = \mathcal{D}_{\mathcal{X}}$$

Thus, if K° is a free spectrahedron, then the containment can be checked via an SDP.

Minimal and maximal matrix Convex sets

Let $C \subset \mathbb{R}^g$ be convex set and assume that $0 \in C$. The minimal matrix convex set generated by C, denoted $\mathcal{W}^{\min}(C)$ is the matrix convex hull of C.

The maximal matrix convex $\mathcal{W}^{\max}(C)$ is the set of $\mathcal{X} \in SM(\mathbb{R})^g$ which satisfy all of the affine linear relations satisfied by C.

In particular, if C is a polyhedron containing 0, then $\mathcal{W}^{\max}(C)$ is a free spectrahedron.

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In particular, if C is a polyhedron containing 0, then $\mathcal{W}^{\mathrm{max}}(C)$ is a free spectrahedron.

Fact: If K is a matrix convex set with $K(1) = C$, then $\mathcal{W}^{\mathrm{min}}(C) \subset K \subset \mathcal{W}^{\mathrm{max}}(C)$

Question: how can one determine the optimal $\gamma \geq 1$ such that

$$\mathcal{W}^{\mathrm{max}}(C) \subset \gamma \mathcal{W}^{\mathrm{min}}(C)$$

Duality of minimal and maximal matrix convex sets

For a compact convex set $C \subset \mathbb{R}^g$, let C' denote its classical dual. That is

$$C' = \{x \in \mathbb{R}^g : \langle x, y \rangle \leq 1 \text{ for all } y \in C\}.$$

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Davidson, Dor-on, Shalit, Solel show that, minimal and maximal matrix convex sets are dual to each other in that

$$(\mathcal{W}^{\text{min}}(C))^\circ = \mathcal{W}^{\text{max}}(C')$$

Furthermore if $0 \in C$, then

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Thus, the free polar dual of a minimal matrix convex set generated by a polyhedron is in fact a free spectrahedron.

Incompatibility of Quantum measurements

Free spectrahedra vs. incompatibility of quantum measurements

Key feature of quantum mechanics is existence of incompatible observables, e.g., position and momentum, which are not jointly measurable.

Idea: Quantify incompatibility of measurements (interpreted as POVMs) by determining the probability that adding noise makes measurements jointly measurable.

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Idea: Quantify incompatibility of measurements (interpreted as POVMs) by determining the probability that adding noise makes measurements jointly measurable.

(Bluhm, Nechita) This is equivalent to determining the smallest constant γ such that

$$\mathcal{D}_{\mathcal{A}}(n) \subset \gamma \mathcal{W}^{\min}(\mathcal{D}_{\mathcal{A}}(1))$$

where $\mathcal{D}_{\mathcal{A}}$ is a Cartesian product of free polyhedron of interest determined by the quantum system and where n is the dimension of the POVM. (E.g, $n = 2$ is qubits).

Incompatibility vs Matrix extreme points

Since $\mathcal{D}_{\mathcal{A}}(n)$ is contained in the matrix convex hull of the matrix extreme points of $\mathcal{D}_{\mathcal{A}}$ of size at most n, one has

$$\mathcal{D}_{\mathcal{A}}(n) \subset \gamma \mathcal{W}^{\min}(\mathcal{D}_{\mathcal{A}}(1))$$

if and only if

$$\mathcal{X} \subset \gamma \mathcal{W}^{\min}(\mathcal{D}_{\mathcal{A}}(1))$$

for all matrix extreme points $\mathcal{X} \in \mathcal{D}_{\mathcal{A}}$ of size at most n.

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for all matrix extreme points $\mathcal{X} \in \mathcal{D}_{\mathcal{A}}$ of size at most n.

This is in turn equivalent to

$$\mathcal{D}_{\mathcal{A}'} \subset \gamma \mathcal{D}_{\mathcal{X}}$$

for all matrix extreme points $\mathcal{X} \in \mathcal{D}_{\mathcal{A}}$ of size at most n, where $\mathcal{D}_{\mathcal{A}'}$ is a free polyhedron whose first level is the classical dual of $\mathcal{D}_{\mathcal{A}}(1)$.

Matrix Extreme points vs Cartesian products

Proposition [Bluhm-E-Klep-Magron-Nechita]

Let $\mathcal{D}_\mathcal{A}$ and $\mathcal{D}_\mathcal{B}$ be bounded free spectrahedra. If $\mathcal{X}$ and $\mathcal{Y}$ are matrix extreme points of $\mathcal{D}_\mathcal{A}$ and $\mathcal{D}_\mathcal{B}$, then $(\mathcal{X}, \mathcal{Y})$ is a matrix extreme points of $\mathcal{D}_\mathcal{A} \times \mathcal{D}_\mathcal{B}$. However, matrix extreme points of $\mathcal{D}_\mathcal{A} \times \mathcal{D}_\mathcal{B}$ need not be pairs of matrix extreme points of $\mathcal{D}_\mathcal{A}$ and $\mathcal{D}_\mathcal{B}$.

If $\mathcal{D}_\mathcal{A}$ is assumed to be a free simplex and $\mathcal{D}_\mathcal{B}$ is the free interval, then $(\mathcal{X}, \mathcal{Y})$ is a real matrix extreme point of $(\mathcal{D}_\mathcal{A} \times \mathcal{D}_\mathcal{B})(2)$ if and only if (up to minor details) $\mathcal{X}$ and $\mathcal{Y}$ are matrix extreme points of $\mathcal{D}_\mathcal{A}$ and $\mathcal{D}_\mathcal{B}$, respectively.

Here $\mathcal{D}_\mathcal{A}$ is a free simplex if $\mathcal{A} \in SM_{g+1}(\mathbb{R})^g$ is a tuple of diagonal matrices and $\mathcal{D}_\mathcal{A}$ is bounded.

In the classical setting, pairs of extreme points are extreme points in a Cartesian product.

The Cartesian product of a free simplex and a line

Theorem [Bluhm-E-Klep-Magron-Nechita]

For $\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}}$ the Cartesian product of a "(real) standard free simplex in k variables" and the "(real) free interval", the smallest constant γ_k such that

$$(\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(2) \subset \gamma_k \mathcal{W}^{\min}((\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(1))$$

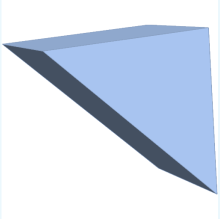
is given by

$$\gamma_k = \frac{2k}{k - 1 + \sqrt{k+1}}$$

We conjecture also that $\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}} \subset \gamma_k \mathcal{W}^{\min}((\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(1))$.

Extreme points of a two variable free simplex and a line

When the simplex has two variables, $\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}}$ at level one is the following spectrahedron,



which has extreme points
$E = \{(1,1,1),(1,-2,1),(-2,1,1),(1,1,-1),(1,-2,-1),(-2,1,-1)\}$. Let $\mathcal{E} \in SM_6(\mathbb{R})^3$ be the diagonal tuple given by taking a direct sum of elements of E.

Fact: $\mathcal{D}_{\mathcal{E}}(1)$ is the classical dual of $(\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(1)$. Thus the optimization in question is equivalent to finding γ s.t.

$$\min_{\gamma \in \mathbb{R}} \quad \text{s.t.} \quad \mathcal{D}_{\mathcal{E}} \subset \gamma \mathcal{D}_{\mathcal{X}}$$

for all matrix extreme points $\mathcal{X}$ of $(\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(2)$.

A feasibility SDP to find γ

Let $E = \{(1,1,1),(1,-2,1),(-2,1,1),(1,1,-1),(1,-2,-1),(-2,1,-1)\}$. Let $\mathcal{E} \in SM_6(\mathbb{R})^3$ be the diagonal tuple given by taking a direct sum of elements of E and let $\mathcal{X} \in (\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(2)$. Then $\mathcal{D}_{\mathcal{E}} \subset \gamma \mathcal{D}_{\mathcal{X}}$ if and only if the HKM SDP

$$
\begin{aligned}
& C \in SM_2(\mathbb{R})^6 \\
& \oplus_{i=1}^{6} C_i \succeq 0, \\
& C_1 - 2C_2 + C_3 + C_4 - 2C_5 + C_6 = X_1, \\
& C_1 + C_2 - 2C_3 + C_4 + C_5 - 2C_6 = X_2, \\
& C_1 + C_2 + C_3 - C_4 - C_5 - C_6 = X_3, \\
& C_1 + C_2 + C_3 + C_4 + C_5 + C_6 = \gamma I,
\end{aligned}
$$

is feasible.

A feasibility SDP to find γ

Theorem [Bluhm-E-Klep-Magron-Nechita]

For $\mathcal{D}_{\mathcal{A}}$ the Cartesian product of a standard free simplex in two variables and interval and $\gamma \in \mathbb{R}$, we have $(\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(2) \subset \gamma\big(\mathrm{co}^{\mathrm{mat}}(\mathcal{D}_{\mathcal{A}}(1))\big)$ if and only if

$$\mathcal{D}_{\mathcal{E}} \subset \mathcal{D}_{\mathcal{X}(\theta)} \qquad \text{for all } \theta \in [0, \pi/2]$$

where

$$\mathcal{X}(\theta) = \left(\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \right)$$

Moreover, if $\gamma_2 = \frac{4}{1+\sqrt{3}}$, then $\mathcal{D}_{\mathcal{E}} \subset \gamma_2 \mathcal{D}_{\mathcal{X}(\theta)}$ for all $\theta \in [0, \pi/2]$.

Proof of the theorem for $k = 2$

Using our classification of the matrix extreme points of $(\mathcal{D}_\mathcal{A} \times \mathcal{D}_\mathcal{B})(2)$, we find that up to unitary equivalence, all matrix extreme points of $(\mathcal{D}_\mathcal{A} \times \mathcal{D}_\mathcal{B})(2)$ have one of the forms

$$\mathcal{X}(\theta) = \left(\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \right)$$

$$\mathcal{Y}(\theta) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \right)$$

$$\mathcal{Z}(\theta) = \left(\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \right)$$

where $\theta \in [0, \pi]$.

Proof of the theorem for $k = 2$

By examining the HKM SDP in question, we can show that solutions for extreme points of the form $\mathcal{X}(\theta)$ give solutions to the remaining forms.

E.g., one can show that $(C_1, C_2, C_3, C_4, C_5, C_6)$ is a solution for $\mathcal{X}(\theta)$ if and only if $(C_2, C_1, C_3, C_5, C_4, C_6)$ is a solution for $\mathcal{Y}(\theta)$.

Arguing similarly allows one to restrict to $\theta \in [0, \pi/2]$.

From here, we construct and exact feasible point of the HKM SDP.

Proof of the theorem for $k \geq 3$

Classifying extreme points of $(\mathcal{D}_\mathcal{A} \times \mathcal{D}_\mathcal{B})(2)$ and examining the form of the HKM SDP allows a dimension reduction from a k-variable simplex to a scaled 2-variable simplex.

Proof of the theorem for $k \geq 3$

Classifying extreme points of $(\mathcal{D}_{\mathcal{A}} \times \mathcal{D}_{\mathcal{B}})(2)$ and examining the form of the HKM SDP allows a dimension reduction from a k-variable simplex to a scaled 2-variable simplex.

This leads to the HKM feasibility SDP

$$
\begin{aligned}
& C \in SM_2(\mathbb{R})^6 \\
& \oplus_{i=1}^{6} C_i \succeq 0, \\
& C_1 - kC_2 + C_3 + C_4 - kC_5 + C_6 = \begin{pmatrix} 1 & 0 \\ 0 & -k \end{pmatrix}, \\
& C_1 + C_2 - kC_3 + C_4 + C_5 - kC_6 = \begin{pmatrix} -k & 0 \\ 0 & 1 \end{pmatrix}, \\
& C_1 + C_2 + C_3 - C_4 - C_5 - C_6 = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -cos(\theta) \end{pmatrix}, \\
& C_1 + C_2 + C_3 + C_4 + C_5 + C_6 = \gamma_k I,
\end{aligned}
$$

A feasible point for $k \geq 3$

The following point is feasible. Set $\alpha(k) := \frac{1}{2k+4\sqrt{k+1}+2}$ and set

$$C_1(\theta) = \alpha(k)\begin{pmatrix} k-1-2\cos(\theta) & (k-1)\sin(\theta) \\ (k-1)\sin(\theta) & k-1+2\cos(\theta) \end{pmatrix}$$

$$C_2(\theta) = \alpha(k)\begin{pmatrix} 1+\cos(\theta) & \left(\sqrt{k+1}+1\right)\sin(\theta) \\ \left(\sqrt{k+1}+1\right)\sin(\theta) & \left(k+2\sqrt{k+1}+2\right)(1-\cos(\theta)) \end{pmatrix}$$

$$C_3(\theta) = \alpha(k)\begin{pmatrix} \left(k+2\sqrt{k+1}+2\right)(1+\cos(\theta)) & \left(\sqrt{k+1}+1\right)\sin(\theta) \\ \left(\sqrt{k+1}+1\right)\sin(\theta) & 1-\cos(\theta) \end{pmatrix}$$

$$C_4(\theta) = \alpha(k)\begin{pmatrix} k-1+2\cos(\theta) & -(k-1)\sin(\theta) \\ -(k-1)\sin(\theta) & k-1-2\cos(\theta) \end{pmatrix}$$

$$C_5(\theta) = \alpha(k)\begin{pmatrix} 1-\cos(\theta) & -\left(\sqrt{k+1}+1\right)\sin(\theta) \\ -\left(\sqrt{k+1}+1\right)\sin(\theta) & \left(k+2\sqrt{k+1}+2\right)(1+\cos(\theta)) \end{pmatrix}$$

$$C_6(\theta) = \alpha(k)\begin{pmatrix} \left(k+2\sqrt{k+1}+2\right)(1-\cos(\theta)) & -\left(\sqrt{k+1}+1\right)\sin(\theta) \\ -\left(\sqrt{k+1}+1\right)\sin(\theta) & 1+\cos(\theta) \end{pmatrix}$$

Checking the point is feasible

To check the point is feasible, one need only put it into the HKM SDP and verify the constraints hold, then verify each $C_j(\theta)$ is positive semidefinite by showing its trace and determinant are nonnegative (which is sufficient since they are 2×2).

This solution does not extend to the $k = 2$ variable case. The issue is that

$$\det(C_1(\theta)) = \det(C_4(\theta)) = \alpha(k)^2(k-3)(k+1)\cos(\theta)^2$$

which is negative when $k = 2$.

The $k = 2$ feasible point we constructed

Set

$$C_1 = \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \qquad \text{and} \qquad C_2 = \alpha\begin{pmatrix}1 & \beta\\ \beta & \beta^2\end{pmatrix}$$

where α and β defined on the next slide. Then (C_1, C_2) is a feasible point of the HKM SDP

$$\begin{aligned}
C_1 \oplus C_2 &\succeq 0,\\
-2C_1 - 2C_2 + X_3(\theta) + \gamma I &\succeq 0,\\
-3C_1 + X_1 + X_2 + \gamma I &\succeq 0,\\
-3C_2 - X_1 + \gamma I &\succeq 0,\\
C_1 + C_2 - X_2/3 - X_3(\theta)/2 - \gamma I/6 &\succeq 0,
\end{aligned}$$

The constraints in the HKM SDP allow us to solve for the remaining variables unknowns in terms of C_1, C_2, so this formulation is equivalent

The choice of α and β

In the previous slide

$$\alpha = \frac{2}{6 + 4\sqrt{3} + \sqrt{3}\beta^2},$$

and

$$\beta_+ := \frac{\zeta_1 + \zeta_2}{\zeta_3} \qquad \text{or} \qquad \beta_- := \frac{\zeta_1 - \zeta_2}{\zeta_3}$$

with

$$\zeta_1 = 12\left(2 + \sqrt{3}\right)\sin(\theta) - 4\sqrt{3}$$

$$\zeta_2 = \sqrt{6}\sqrt{8\eta_1 \sin(\theta) + 6\eta_1 \sin(2\theta) + 6\eta_2 \cos(\theta) + 6\left(2 + \sqrt{3}\right)\cos(2\theta) + 181\sqrt{3} + 318}$$

$$\zeta_3 = -6\sin(\theta) + 12\left(2 + \sqrt{3}\right)\cos(\theta) + 14\sqrt{3} + 21$$

and

$$\eta_1 = 12 + 7\sqrt{3} \qquad \text{and} \qquad \eta_2 = 54 + 31\sqrt{3}.$$

For context as to how much messier this is than the $k \geq 3$ case...

A big challenge ends up being showing that $C_1 + C_2 - X_2/3 - X_3(\theta)/2 - \gamma I/6 \succeq 0$. One can do this by looking at the trace and determinant. We couldn't show that this is positive if one fixes a choice β_+ of β_-.

We showed that if $h(\theta)$ defined below is positive for $\theta \in [0, \pi/2]$, then for each θ, either choosing β_+ or β_- will work.

$$
\begin{aligned}
h(\theta) \;=\;& 3\left(1659159 + 957244\sqrt{3}\right)\sin(\theta) + 24\left(108048 + 62413\sqrt{3}\right)\sin(2\theta) \\
& -18\left(84547 + 48802\sqrt{3}\right)\sin(3\theta) - 36\left(48096 + 27769\sqrt{3}\right)\sin(4\theta) \\
& -81\left(5307 + 3064\sqrt{3}\right)\sin(5\theta) + 48\left(11401 + 6598\sqrt{3}\right)\cos(\theta) \\
& +54\left(5341 + 3100\sqrt{3}\right)\cos(2\theta) - 36\left(7538 + 4359\sqrt{3}\right)\cos(3\theta) \\
& -108\left(3469 + 2003\sqrt{3}\right)\cos(4\theta) - 324\left(362 + 209\sqrt{3}\right)\cos(5\theta) \\
& +62\left(3963 + 2266\sqrt{3}\right).
\end{aligned}
$$