Real vs complex free spectrahedra: The evils of the reals and the crimes of the complexes

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Convex Combinations

Given a set $C \subset \mathbb{R}^g$ and a finite collection of tuples $\{x^\ell\} \subset C$ where $x^\ell = (x_1^\ell, x_2^\ell, \ldots, x_g^\ell)$ and coefficients $\alpha_\ell \geq 0$ a convex combination is a sum of the form

$$\sum_{\ell=1}^{k} \alpha_\ell x^\ell \in \mathbb{R}^g \qquad \text{such that} \qquad \sum_{\ell=1}^{k} \alpha_\ell = 1$$

The convex hull of a set C is the set of all convex combinations of C. Say C is convex if it is closed under convex combinations.

A point $x \in C$ is an extreme point of C if it cannot be expressed as a nontrivial convex combination of elements of C.

Convex sets have many nice properties

Theorem [Carathéodory (also see Krein-Milman)]

Let $C \subset \mathbb{R}^g$ be a closed bounded convex set. Then C is the convex hull of its extreme points.

Furthermore, every element of C can be expressed as a convex combination of at most $g + 1$ extreme points of C.

Linear matrix inequalities give convex sets

A (monic) linear pencil is a matrix valued function $L_{\mathcal{A}}$ of the form

$$L_{\mathcal{A}}(x) := \mathbf{I}_d - \sum_{j=1}^{g} \mathbf{A}_j x_j = I_d - \Lambda_{\mathcal{A}}(x),$$

where $\mathcal{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_g)$ with each $\mathbf{A}_j$ symmetric $d \times d$ and $x = (x_1, \ldots, x_g) \in \mathbb{R}^g$

A Linear Matrix Inequality (LMI) is one of the form:

$$L_{\mathcal{A}}(x) \succeq 0, \qquad i.e., \qquad L_{\mathcal{A}}(x) \text{ is positive semidefinite.}$$

The set of solutions x above is a convex set called a spectrahedron. Spectrahedra are the feasibility domains of convex optimization problems called semidefinite programs (SDP).

Spectrahedron example

Take $\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$L_{\mathcal{A}}(x) = \mathbf{I}_2 - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x_1 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_2 = \begin{pmatrix} 1 - x_1 & -x_2 \\ -x_2 & 1 + x_1 \end{pmatrix}$$

Observe $L_{\mathcal{A}}(x) \succeq 0$ IFF $\det(L_{\mathcal{A}}(x)) = 1 - x_1^2 - x_2^2 \geq 0$. So $L_{\mathcal{A}}(x)$ defines circle in $\mathbb{R}^2$.

Dimension free sets

Let $SM_n(\mathbb{F})^g$ denote g-tuples of self-adjoint $n \times n$ matrices over $\mathbb{F}$. I.e. if $\mathcal{X} \in SM_n(\mathbb{F})^g$ then

$$\mathcal{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_g)$$

where each $\mathbf{X}_i$ is a self-adjoint $n \times n$ matrix.

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Our goal: Study solution sets of linear matrix inequalities over $SM(\mathbb{F})^g$.

We will see $\mathbb{F} = \mathbb{R}$ vs $\mathbb{F} = \mathbb{C}$ can make a big difference.

Free Linear matrix inequalities

A free (monic) linear pencil is a matrix valued function $L_{\mathcal{A}}$ of the form

$$L_{\mathcal{A}}(\mathcal{X}) := \mathbf{I}_{dn} - \sum_{j=1}^{g} \mathbf{A}_j \otimes \mathbf{X}_j = \mathbf{I}_{dn} - \Lambda_{\mathcal{A}}(\mathcal{X}),$$

where $\mathcal{A} \in SM_d(\mathbb{F})^g$ and $\mathcal{X} \in SM_n(\mathbb{F})^g$. Here $\otimes$ denotes the Kronecker Product. E.g.

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 2\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} & 3\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} \\ 3\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} & 4\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix} \end{pmatrix}$$

A Free Linear Matrix Inequality (LMI) is one of the form:

$$L_{\mathcal{A}}(\mathcal{X}) \succeq 0.$$

Free spectrahedra

For each fixed n the solution set

$$\mathcal{D}_{\mathcal{A}}(n) = \{\mathcal{X} \in SM_n(\mathbb{R})^g : L_{\mathcal{A}}(\mathcal{X}) = \mathbf{I}_{dn} - \sum_{j=1}^{g} \mathbf{A}_j \otimes \mathbf{X}_j \succeq 0\}$$

is called a free spectrahedron at level n.

The set $\mathcal{D}_{\mathcal{A}} = \cup_n \mathcal{D}_{\mathcal{A}}(n) \subset \cup_n SM_n(\mathbb{R})^g$ is called a free spectrahedron.

If $\mathcal{A}$ a tuple of simultaneously diagonalizable matrices, then $\mathcal{D}_{\mathcal{A}}$ is called free polyhedron.

Matrix Convex Combinations

Given a finite collection of tuples $\{\mathcal{X}^\ell\} \subset SM(\mathbb{F})^g$ where $\mathcal{X}^\ell = (\mathbf{X}_1^\ell, \dots, \mathbf{X}_g^\ell) \in SM_{n_\ell}(\mathbb{F})^g$, a matrix convex combination is a sum of the form

$$\sum_{\ell=1}^{k} \mathbf{V}_\ell^* \mathcal{X}^\ell \mathbf{V}_\ell \in SM_n(\mathbb{R})^g \qquad \text{such that} \qquad \sum_{\ell=1}^{k} \mathbf{V}_\ell^* \mathbf{V}_\ell = \mathbf{I}_n$$

Here the $\mathbf{V}_\ell$ are $n_\ell \times n$ matrices which serve as convex coefficients, and

$$\mathbf{V}_\ell^* \mathcal{X}^\ell \mathbf{V}_\ell = (\mathbf{V}_\ell^* \mathbf{X}_1^\ell \mathbf{V}_\ell, \dots, \mathbf{V}_\ell^* \mathbf{X}_g^\ell \mathbf{V}_\ell).$$

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For $K \subset SM(\mathbb{F})^g$ let $\mathrm{co}^{\mathrm{mat}}(K)$ denote the set of matrix convex combinations of K. Say K is matrix convex if it is closed under matrix convex combinations, i.e., if $K = \mathrm{co}^{\mathrm{mat}}(K)$.

Say K is bounded if there exists a $M \geq 0$ such that $M\mathbf{I} - \sum_{i=1}^{g} \mathbf{X}_i^2 \succeq 0$ for all $\mathcal{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_g) \in K$.

Matrix convex combinations allow for convex combinations of tuples of different sizes

For example, if $\mathcal{X}^1 \in SM_{n_1}(\mathbb{F})^g$ and $\mathcal{X}^2 \in SM_{n_2}(\mathbb{F})^g$ and

$$\mathbf{V}_1^* = (\mathbf{I}_{n_1} \quad \mathbf{0}_{n_1 \times n_2}) \qquad \text{and} \qquad \mathbf{V}_2^* = (\mathbf{0}_{n_2 \times n_1} \quad \mathbf{I}_{n_2}),$$

then

$$\mathbf{V}_1^* \mathcal{X}^1 \mathbf{V}_1 + \mathbf{V}_2^* \mathcal{X}^2 \mathbf{V}_2 = \mathcal{X}^1 \oplus \mathcal{X}^2 = \begin{pmatrix} \mathcal{X}^1 & 0 \\ 0 & \mathcal{X}^2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{V}_1^* \mathbf{V}_1 + \mathbf{V}_2^* \mathbf{V}_2 = \mathbf{I}_{n_1+n_2}.$$

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then

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On the other hand, if $\mathcal{X} \in SM_n(\mathbb{R})^g$, and $\mathbf{V} \in \mathbb{F}^{n \times m}$ and $\mathbf{V}^*\mathbf{V} = \mathbf{I}_m$, then

$$\mathbf{V}^* \mathcal{X} \mathbf{V} \in SM_m(\mathbb{R})^g$$

is a matrix convex combination of $\mathcal{X}$.

Extreme points of matrix convex sets

Say $\mathcal{X}$ is a free (absolute) extreme point of $K \subset SM(\mathbb{F})^g$ if $\mathcal{X}$ cannot be expressed as a nontrivial matrix convex combination of any elements of K.

If $E \subset K$ and $\mathrm{co}^{\mathrm{mat}}(E) = K$, then E must contain the free extreme points of K (up to unitary equivalence).

If $\mathcal{D}_\mathcal{A} \subset SM(\mathbb{R})^g$ is a real free spectrahedron, the the classical extreme points of $\mathcal{D}_\mathcal{A}(1)$ are free extreme points of $\mathcal{D}_\mathcal{A}$. The same is not true over $\mathbb{C}$

Free extreme points vs Free spectrahedra. The crimes of the complexes

Theorem [E-Helton 19]

Let $\mathcal{D}_{\mathcal{A}} \subset SM(\mathbb{R})^g$ be a bounded real free spectrahedron and let $\mathcal{X} \in \mathcal{D}_{\mathcal{A}}(n)$. Then $\mathcal{X}$ is a free convex combination of free extreme points of $\mathcal{D}_{\mathcal{A}}$ whose sum of sizes is at most $n(g+1)$.

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Theorem [Passer 22]

There exist closed bounded complex free spectrahedra that have no free extreme points.

Complex conjugation closed as the solution?

Theorem [E-Helton 19]

Let $\mathcal{D}_{\mathcal{A}} \subset SM(\mathbb{C})^g$ be a bounded complex free spectrahedron and let $\mathcal{X} \in \mathcal{D}_{\mathcal{A}}(n)$. Additionally assume $\mathcal{D}_{\mathcal{A}}$ is closed under complex conjugation. Then $\mathcal{X}$ is a free convex combination of free extreme points of $\mathcal{D}_{\mathcal{A}}$ whose sum of sizes is at most $2n(g+1)$.

Complex conjugation closed is equivalent to assuming there exists a real matrix tuple $\mathcal{B}$ so that $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{B}}$.

Complex conjugation closed is not necessary for free extreme to span a complex free spectrahedron.

Free spectrahedrops, polar duals, and real problems

Free spectrahedrops a.k.a free spectrahedra shadows

Let $\mathcal{A} \in SM_d(\mathbb{R})^g$ and fix $h < g$. The free spectrahedrop $\mathrm{proj}_h \mathcal{D}_\mathcal{A}$ is the coordinate projection of the free spectrahedron $\mathcal{D}_\mathcal{A}$ onto its first h variables.

That is

$$\mathrm{proj}_h \mathcal{D}_\mathcal{A} := \{\mathcal{X} \in SM(\mathbb{R})^h : \exists\ \mathcal{Y} \text{ s.t. } L_\mathcal{A}(\mathcal{X}, \mathcal{Y}) \succeq 0\}.$$

Question: Is a closed bounded free spectrahedrop that is closed under complex conjugation the matrix convex hull of its free extreme points. We will prove: NO!

Free polar duals and free spectrahedrops

The (classical) polar dual $C^\bullet$ of a convex set C is

$$C^\bullet = \{y \in \mathbb{R}^g : 0 \leq 1 - \langle x, y\rangle = \ L_y(x) \text{ for all } x \in C\} = \cap_{x \in C} \mathcal{D}_x(1)$$

The free polar dual K° of a matrix convex set $K \subset SM(\mathbb{F})^g$ is

$$K^\circ := \{\mathcal{Y} \in SM(\mathbb{F})^g : \ L_{\mathcal{X}}(\mathcal{Y}) \succeq 0 \text{ for all } \mathcal{X} \in K\} = \cap_{\mathcal{X} \in K} \mathcal{D}_{\mathcal{X}}$$

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Theorem [Helton-Klep-McCullough 17]

Let $\mathcal{D}_{\mathcal{A}}$ be a closed bounded free spectrahedron. Then $\mathcal{D}_{\mathcal{A}}^\circ = \mathrm{co}^{\mathrm{mat}}(A)$. Moreover, $\mathcal{D}_{\mathcal{A}}^\circ$ is the projection of a closed bounded free spectrahedron. Furthermore, a spectrahdrop representation can be computed algorithmically.

Spectrahedrop example

Recall that if $\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\mathcal{D}_{\mathcal{A}}(1) = \overline{\mathbb{D}}$.

In fact, one can show $\mathcal{D}_{\mathcal{A}} = \mathrm{co}^{\mathrm{mat}}(\mathcal{D}_{\mathcal{A}}(1))$, thus $\mathcal{D}_{\mathcal{A}}$ is the minimal matrix convex set whose first level is $\overline{\mathbb{D}}$.

As a consequence, its free polar dual $\mathcal{D}_{\mathcal{A}}^{\circ} = \mathrm{co}^{\mathrm{mat}}\mathcal{A}$ is the maximal matrix convex set whose first level is $\overline{\mathbb{D}}$. Furthermore, by HKM, $\mathrm{co}^{\mathrm{mat}}\mathcal{A}$ is a free spectrahedrop.

Spectrahedrop example

Using HKM, one computes that if $\mathbf{P}_1 = \mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{P}_2 = \mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{P}_3 = \mathbf{A}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, then $\mathcal{D}_{\mathcal{A}} = \mathrm{proj}_2 \mathcal{D}_{\mathcal{P}}$.

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One can verify this by checking that

$$L_{\mathcal{P}}(\mathcal{P}) \succeq 0 \qquad \text{thus} \qquad \mathcal{A} \in \mathrm{proj}_2 \mathcal{D}_{\mathcal{P}} \qquad \text{and} \qquad \mathrm{co}^{\mathrm{mat}}(\mathcal{A}) \in \mathrm{proj}_2 \mathcal{D}_{\mathcal{P}}.$$

Additionally, $\mathrm{proj}_2 \mathcal{D}_{\mathcal{P}}(1) = \overline{\mathbb{D}}$, so maximality of $\mathrm{co}^{\mathrm{mat}}(\mathcal{A})$ gives

$$\mathrm{proj}_2 \mathcal{D}_{\mathcal{P}} \subset \mathrm{co}^{\mathrm{mat}}(\mathcal{A}).$$

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$$\mathrm{proj}_2 \mathcal{D}_{\mathcal{P}} \subset \mathrm{co}^{\mathrm{mat}}(\mathcal{A}).$$

$\mathcal{D}_{\mathcal{P}}$ is not closed under complex conjugation! One has $\overline{\mathcal{P}} \notin \mathcal{D}_{\mathcal{P}}$.

Level 1 extreme points of real free spectrahedrops

Theorem [E Passer (in progress)]

Let $\mathcal{D}_\mathcal{A}$ *be a bounded free spectrahedron that is closed under complex conjugation and let* $\mathrm{proj}_h \mathcal{D}_\mathcal{A}$ *be a spectrahedrop defined by* $\mathcal{D}_\mathcal{A}$*. Then the classical extreme points of* $\mathrm{proj}_h \mathcal{D}_\mathcal{A}(1)$ *are free extreme points.*

Level 1 extreme points of real free spectrahedrops

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In the case of $\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have seen $\mathcal{D}_\mathcal{A}^\circ = \mathrm{co}^{\mathrm{mat}} \mathcal{A}$.

Up to unitary equivalence $\mathcal{A}$ is the only free extreme point of $\mathcal{D}_\mathcal{A}^\circ$, so it has no level 1 extreme points!

Polar duals of real free spectrahedra. The evils of the reals.

Theorem [E Passer (in progress)]

The polar dual of a real free spectrahedron is rarely a real free spectrahedrop.

If $\mathcal{D}_\mathcal{A}$ is a real free spectrahedron and $\mathcal{D}^\circ_\mathcal{A}$ is the projection of a real free spectrahedron, then $\mathcal{D}_\mathcal{A}(1)$ must be a polyhedron, and $\mathcal{A}$ must be unitarily equivalent to a tuple $\mathcal{B} \oplus \mathcal{C}$ where $\mathcal{B}$ is a g-tuple of $g+1 \times g+1$ diagonal matrices.

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If $\mathcal{D}_{\mathcal{A}}$ is a real free spectrahedron and $\mathcal{D}_{\mathcal{A}}^{\circ}$ is the projection of a real free spectrahedron, then $\mathcal{D}_{\mathcal{A}}(1)$ must be a polyhedron, and $\mathcal{A}$ must be unitarily equivalent to a tuple $\mathcal{B} \oplus \mathcal{C}$ where $\mathcal{B}$ is a g-tuple of $g+1 \times g+1$ diagonal matrices.

Proof sketch: The free extreme points of $\mathcal{D}_{\mathcal{A}}^{\circ} = \mathrm{co}^{\mathrm{mat}}(\mathcal{A})$ are (up to unitary equivalence) the irreducible direct summands of $\mathcal{A}$. Thus $\mathcal{D}_{\mathcal{A}}^{\circ}$ has only finitely many free extreme points. On the other hand, if $\mathcal{D}_{\mathcal{A}}^{\circ}$ is the projection of a real free spectrahedron, then it must have at least $g+1$ extreme points at level 1.

Polar duals of free polyhedra are projections of real free spectrahedra

Theorem [E Passer (in progress)]

Let $\mathcal{A}$ be a tuple of real diagonal matrices and assume that $\mathcal{D}_{\mathcal{A}}$ is bounded. Then the real matrix convex set $\mathcal{D}_{\mathcal{A}}^{\circ}$ is the projection of a real free spectrahedron

The proof essentially follows from the fact that a polyhedron can be written as the projection of a simplex.

The projection constructed from our proof is different from projection constructed by the HKM algorithm.

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The proof essentially follows from the fact that a polyhedron can be written as the projection of a simplex.

The projection constructed from our proof is different from projection constructed by the HKM algorithm.

Question: The free spectrahedra appearing in the theorem statement are maximal free polyhedra. Must $\mathcal{D}_{\mathcal{A}}$ be a maximal free polyhedron if $\mathcal{D}_{\mathcal{A}}^{\circ}$ is the projection of a real free spectrahedron?

Free spectrahedrops without free extreme points

Theorem [E Passer (in progress)]

There exists a free spectrahedrop that is closed under complex conjugation but has no free extreme points.

Free spectrahedrops without free extreme points

Theorem [E Passer (in progress)]

There exists a free spectrahedrop that is closed under complex conjugation but has no free extreme points.

Proof sketch: Let $\mathcal{D}_{\mathcal{A}}$ be a closed complex free spectrahedron that has no free extreme points.

Using Hartz-Lupini 21, $K := \mathrm{co}^{\mathrm{mat}}(\mathcal{D}_{\mathcal{A}} \cup \overline{\mathcal{D}_{\mathcal{A}}})$ is a closed bounded matrix convex set. Using, Helton-Klep-McCullough 17, K is also a free spectrahedrop.

K is closed under complex conjugation. Additionally, K the free extreme points of K must be extreme points of free extreme points of $\mathcal{D}_{\mathcal{A}}$ or $\overline{\mathcal{D}_{\mathcal{A}}}$. But neither set has free extreme points!